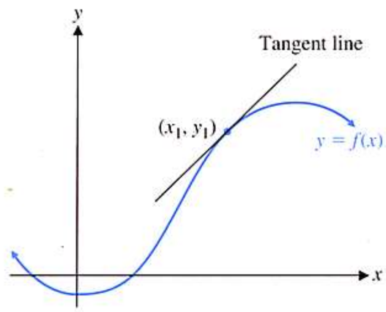
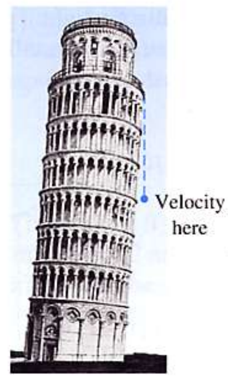
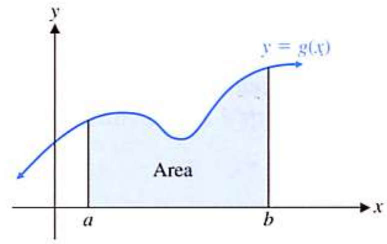



(A) Find the equation of the tangent line at $(x_1, y_1)$, given $y = f(x)$




(B) Find the instantaneous velocity of a falling object




(C) Find the indicated area bounded by $y = g(x)$, $x = a$, $x = b$, and the $x$ axis

# Section 10-1 INTRODUCTION TO LIMITS

- Functions and Graphs: Brief Review
- Limits: A Graphical Approach
- Limits: An Algebraic Approach
- Limits of Difference Quotients

Basic to the study of calculus is the concept of a *limit.* This concept helps us to describe, in a precise way, the behavior of $f(x)$ when $x$ is close, but not equal, to a particular value $c$. In this section, we develop an intuitive and informal approach to evaluating limits. Our discussion concentrates on developing and understanding concepts rather than on presenting formal mathematical details.

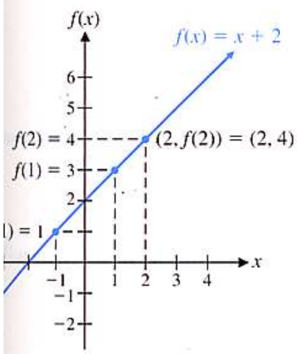


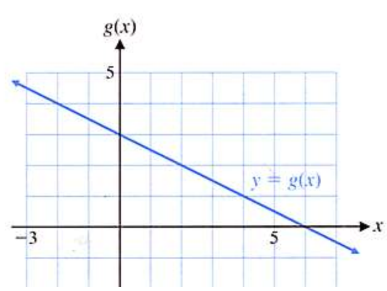

**FIGURE 1**

## Functions and Graphs: Brief Review

The graph of the function $y = f(x) = x + 2$ is the graph of the set of all ordered pairs $(x, f(x))$. For example, if $x = 2$, then $f(2) = 4$ and $(2, f(2)) = (2, 4)$ is a point on the graph of $f$. Figure 1 shows $(-1, f(-1))$, $(1, f(1))$, and $(2, f(2))$ plotted on the graph of $f$. Notice that the domain values $-1$, 1, and 2 are associated with the $x$ axis and the range values $f(-1) = 1$, $f(1) = 3$, and $f(2) = 4$ are associated with the $y$ axis.

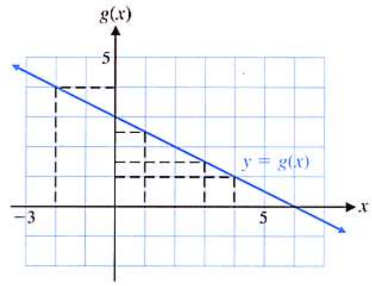
Given $x$, it is sometimes useful to be able to read $f(x)$ directly from the graph of $f$. Example 1 reviews this process.

**EXAMPLE 1** **Finding Values of a Function from Its Graph** Complete the following table, using the given graph of the function $g$:



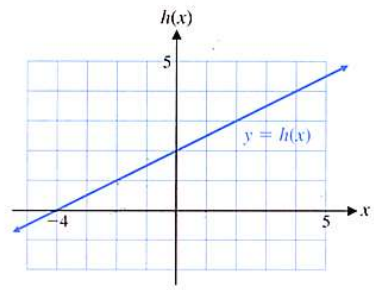

| $x$ | $g(x)$ |
|---|---|
| $-2$ | |
| 1 | |
| 3 | |
| 4 | |

**SOLUTION** To determine $g(x)$, proceed vertically from the $x$ value on the $x$ axis to the graph of $g$ and then horizontally to the corresponding $y$ value $g(x)$ on the $y$ axis (as indicated by the dashed lines):




| $x$ | $g(x)$ |
|---|---|
| −2 | 4.0 |
| 1 | 2.5 |
| 3 | 1.5 |
| 4 | 1.0 |

**MATCHED PROBLEM 1** Complete the following table, using the given graph of the function $h$:




| $x$ | $h(x)$ |
|---|---|
| −2 | |
| −1 | |
| 0 | |
| 1 | |
| 2 | |
| 3 | |
| 4 | |

## Limits: A Graphical Approach

We introduce the important concept of a *limit* through an example, which will lead to an intuitive definition of the concept.

**EXAMPLE 2** **Analyzing a Limit** Let $f(x) = x + 2$. Discuss the behavior of the values of $f(x)$ when $x$ is close to 2.

**SOLUTION** We begin by drawing a graph of $f$ that includes the domain value $x = 2$ (Fig. 2).

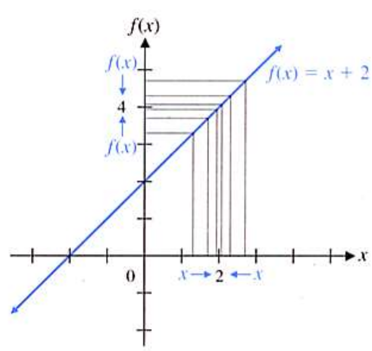



FIGURE 2

In Figure 2, we are using a static drawing to describe a dynamic process. This requires careful interpretation. The thin vertical lines in Figure 2 represent values of $x$ that are close to 2. The corresponding horizontal lines identify the value of $f(x)$ associated with each value of $x$. [Example 1 dealt with the relationship between $x$ and $f(x)$ on a graph.] The graph in Figure 2 indicates that as the values of $x$ get closer and closer to 2 on either side of 2, the corresponding values of $f(x)$ get closer and closer to 4. Symbolically, we write

$$\lim_{x \to 2} f(x) = 4$$

This equation is read as "The limit of $f(x)$ as $x$ approaches 2 is 4." Note that $f(2) = 4$. That is, the value of the function at 2 and the limit of the function as $x$ approaches 2 are the same. This relationship can be expressed as

$$\lim_{x \to 2} f(x) = f(2) = 4$$

Graphically, this means that there is no hole or break in the graph of $f$ at $x = 2$. ▬

**MATCHED PROBLEM 2** Let $f(x) = x + 1$. Discuss the behavior of the values of $f(x)$ when $x$ is close to 1.

We now present an informal definition of the important concept of a limit. A precise definition is not needed for our discussion, but one is given in a footnote.*

**DEFINITION Limit**

We write

$$\lim_{x \to c} f(x) = L \quad \text{or} \quad f(x) \to L \quad \text{as} \quad x \to c$$

if the functional value $f(x)$ is close to the single real number $L$ whenever $x$ is close, but not equal, to $c$ (on either side of $c$).

**Note:** The existence of a limit at $c$ has nothing to do with the value of the function at $c$. In fact, $c$ may not even be in the domain of $f$. However, the function must be defined on both sides of $c$.

The next example involves the **absolute value function:**

$$f(x) = |x| = \begin{cases} -x & \text{if } x < 0 \\ x & \text{if } x \geq 0 \end{cases} \qquad \begin{matrix} f(-2) = |-2| = -(-2) = 2 \\ f(3) = |3| = 3 \end{matrix}$$

The graph of $f$ is shown in Figure 3.

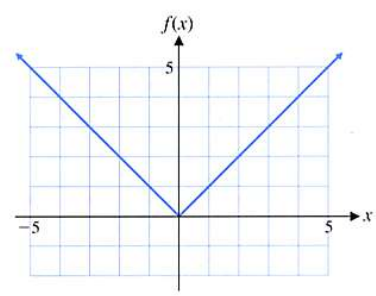



FIGURE 3 $f(x) = |x|$

* To make the informal definition of *limit* precise, the use of the word *close* must be made more precise. This is done as follows: We write $\lim_{x \to c} f(x) = L$ if, for each $e > 0$, there exists a $d > 0$ such that $|f(x) - L| < e$ whenever $0 < |x - c| < d$. This definition is used to establish particular limits and to prove many useful properties of limits that will be helpful to us in finding particular limits. [Even though intuitive notions of a limit existed for a long time, it was not until the nineteenth century that a precise definition was given by the German mathematician Karl Weierstrass (1815–1897).]

**EXAMPLE 3** **Analyzing a Limit** Let $h(x) = |x|/x$. Explore the behavior of $h(x)$ for $x$ near, but not equal, to 0. Find $\lim_{x \to 0} h(x)$ if it exists.

**SOLUTION** The function $h$ is defined for all real numbers except 0. For example,

$$h(-2) = \frac{|-2|}{-2} = \frac{2}{-2} = -1$$

$$h(0) = \frac{|0|}{0} = \frac{0}{0} \qquad \text{Not defined}$$

$$h(2) = \frac{|2|}{2} = \frac{2}{2} = 1$$

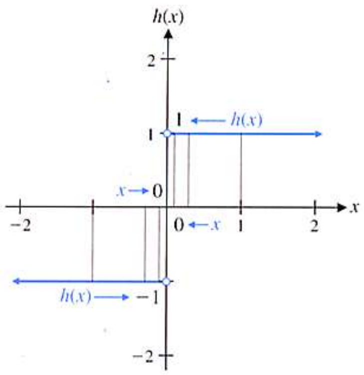


**FIGURE 4**

In general, $h(x)$ is $-1$ for all negative $x$ and 1 for all positive $x$. Figure 4 illustrates the behavior of $h(x)$ for $x$ near 0. Note that the absence of a solid dot on the vertical axis indicates that $h$ is not defined when $x = 0$.

When $x$ is near 0 (on either side of 0), is $h(x)$ near one specific number? The answer is "No," because $h(x)$ is $-1$ for $x < 0$ and 1 for $x > 0$. Consequently, we say that

$$\lim_{x \to 0} \frac{|x|}{x} \text{ does not exist}$$

Thus, neither $h(x)$ nor the limit of $h(x)$ exists at $x = 0$. However, the limit from the left and the limit from the right both exist at 0, but they are not equal. ∎

**MATCHED PROBLEM 3** Graph

$$h(x) = \frac{x - 2}{|x - 2|}$$

and find $\lim_{x \to 2} h(x)$ if it exists.

In Example 3, we saw that the values of the function $h(x)$ approached two different numbers, depending on the direction of approach, and it was natural to refer to these values as "the limit from the left" and "the limit from the right." These experiences suggest that the notion of **one-sided limits** will be very useful in discussing basic limit concepts.

**DEFINITION** **One-Sided Limits**

We write

$$\lim_{x \to c^-} f(x) = K \qquad x \to c^- \text{ is read "}x\text{ approaches }c\text{ from the left" and means } x \to c \text{ and } x < c.$$

and call $K$ the **limit from the left** or the **left-hand limit** if $f(x)$ is close to $K$ whenever $x$ is close to, but to the left of, $c$ on the real number line. We write

$$\lim_{x \to c^+} f(x) = L \qquad x \to c^+ \text{ is read "}x\text{ approaches }c\text{ from the right" and means } x \to c \text{ and } x > c.$$

and call $L$ the **limit from the right** or the **right-hand limit** if $f(x)$ is close to $L$ whenever $x$ is close to, but to the right of, $c$ on the real number line.

If no direction is specified in a limit statement, we will always assume that the limit is **two sided** or **unrestricted.** Theorem 1 states an important relationship between one-sided limits and unrestricted limits.

### THEOREM 1 ON THE EXISTENCE OF A LIMIT

For a (two-sided) limit to exist, the limit from the left and the limit from the right must exist and be equal. That is,

$$\lim_{x\to c} f(x) = L \quad \text{if and only if} \quad \lim_{x\to c^-} f(x) = \lim_{x\to c^+} f(x) = L$$

In Example 3,

$$\lim_{x\to 0^-} \frac{|x|}{x} = -1 \qquad \text{and} \qquad \lim_{x\to 0^+} \frac{|x|}{x} = 1$$

Since the left- and right-hand limits are *not* the same,

$$\lim_{x\to 0} \frac{|x|}{x} \text{ does not exist}$$

**EXAMPLE 4** **Analyzing Limits Graphically** Given the graph of the function $f$ shown in Figure 5, discuss the behavior of $f(x)$ for $x$ near (A) $-1$, (B) 1, and (C) 2.

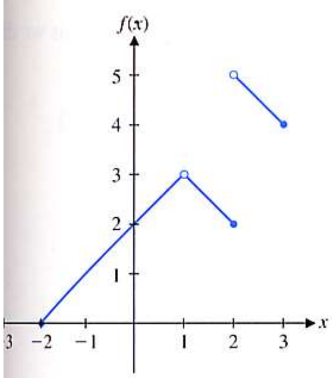
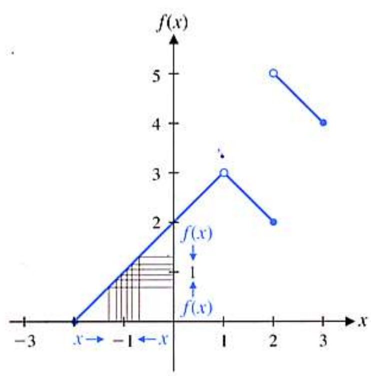


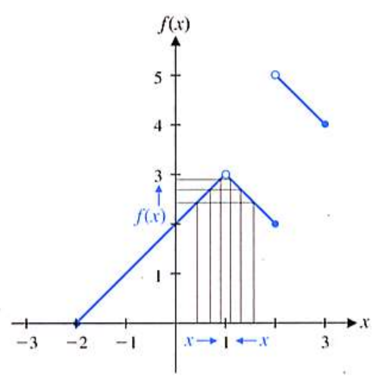

FIGURE 5

**SOLUTION** (A) Since we have only a graph to work with, we use vertical and horizontal lines to relate the values of $x$ and the corresponding values of $f(x)$. For any $x$ near $-1$ on either side of $-1$, we see that the corresponding value of $f(x)$, determined by a horizontal line, is close to 1.



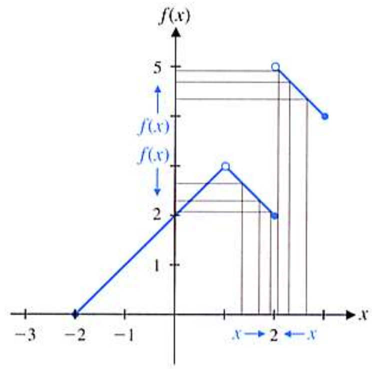

$$\lim_{x\to -1^-} f(x) = 1$$
$$\lim_{x\to -1^+} f(x) = 1$$
$$\lim_{x\to -1} f(x) = 1$$
$$f(-1) = 1$$

(B) Again, for any $x$ near, but not equal to, 1, the vertical and horizontal lines indicate that the corresponding value of $f(x)$ is close to 3. The open dot at (1, 3), together with the absence of a solid dot anywhere on the vertical line through $x = 1$, indicates that $f(1)$ is not defined.




$$\lim_{x\to 1^-} f(x) = 3$$
$$\lim_{x\to 1^+} f(x) = 3$$
$$\lim_{x\to 1} f(x) = 3$$
$f(1)$ not defined

(C) The abrupt break in the graph at $x = 2$ indicates that the behavior of the graph near $x = 2$ is more complicated than in the two preceding cases. If $x$ is close to 2 on the left side of 2, the corresponding horizontal line intersects the $y$ axis at a point close to 2. If $x$ is close to 2 on the right side of 2, the corresponding horizontal line intersects the $y$ axis at a point close to 5. Thus, this is a case where the one-sided limits are different.




$$\lim_{x \to 2^-} f(x) = 2$$
$$\lim_{x \to 2^+} f(x) = 5$$
$$\lim_{x \to 2} f(x) \text{ does not exist}$$
$$f(2) = 2$$

**MATCHED PROBLEM 4** Given the graph of the function $f$ shown in Figure 6, discuss the following, as we did in Example 4:

(A) Behavior of $f(x)$ for $x$ near 0
(B) Behavior of $f(x)$ for $x$ near 1
(C) Behavior of $f(x)$ for $x$ near 3

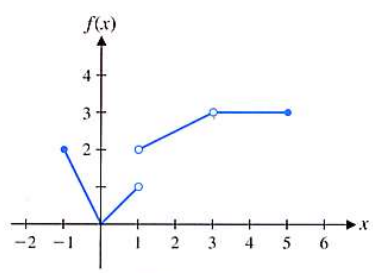



FIGURE 6

**INSIGHT**

In Example 4B, note that $\lim_{x \to 1} f(x)$ exists even though $f$ is not defined at $x = 1$ and the graph has a hole at $x = 1$. In general, the value of a function at $x = c$ has no effect on the limit of the function as $x$ approaches $c$.

## Limits: An Algebraic Approach

Graphs are very useful tools for investigating limits, especially if something unusual happens at the point in question. However, many of the limits encountered in calculus are routine and can be evaluated quickly with a little algebraic simplification, some intuition, and basic properties of limits. The following list of properties of limits forms the basis for this approach:

## THEOREM 2 PROPERTIES OF LIMITS

Let $f$ and $g$ be two functions, and assume that

$$\lim_{x \to c} f(x) = L \qquad \lim_{x \to c} g(x) = M$$

where $L$ and $M$ are real numbers (both limits exist). Then

1. $\lim_{x \to c} k = k$ for any constant $k$
2. $\lim_{x \to c} x = c$
3. $\lim_{x \to c}[f(x) + g(x)] = \lim_{x \to c} f(x) + \lim_{x \to c} g(x) = L + M$
4. $\lim_{x \to c}[f(x) - g(x)] = \lim_{x \to c} f(x) - \lim_{x \to c} g(x) = L - M$
5. $\lim_{x \to c} kf(x) = k \lim_{x \to c} f(x) = kL$ for any constant $k$
6. $\lim_{x \to c}[f(x) \cdot g(x)] = [\lim_{x \to c} f(x)][\lim_{x \to c} g(x)] = LM$
7. $\lim_{x \to c} \dfrac{f(x)}{g(x)} = \dfrac{\lim_{x \to c} f(x)}{\lim_{x \to c} g(x)} = \dfrac{L}{M}$ if $M \neq 0$
8. $\lim_{x \to c} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x \to c} f(x)} = \sqrt[n]{L}$ $L > 0$ for $n$ even

Each property in Theorem 2 is also valid if $x \to c$ is replaced everywhere by $x \to c^-$ or replaced everywhere by $x \to c^+$.

*Explore & Discuss* **1**

The properties listed in Theorem 2 can be paraphrased in brief verbal statements. For example, property 3 simply states that *the limit of a sum is equal to the sum of the limits.* Write brief verbal statements for the remaining properties in Theorem 2.

**EXAMPLE 5** **Using Limit Properties** Find $\lim_{x \to 3}(x^2 - 4x)$.

**SOLUTION**

$$\lim_{x \to 3}(x^2 - 4x) = \lim_{x \to 3} x^2 - \lim_{x \to 3} 4x \qquad \text{Property 4}$$

$$= \left(\lim_{x \to 3} x\right) \cdot \left(\lim_{x \to 3} x\right) - 4 \lim_{x \to 3} x \qquad \text{Properties 5 and 6}$$

$$= 3 \cdot 3 - 4 \cdot 3 = -3 \qquad \text{Property 2}$$

With a little practice, you will soon be able to omit the steps in the dashed boxes and simply write

$$\lim_{x \to 3}(x^2 - 4x) = 3 \cdot 3 - 4 \cdot 3 = -3$$

**MATCHED PROBLEM 5** Find $\lim_{x \to -2}(x^2 + 5x)$.

What happens if we try to evaluate a limit like the one in Example 5, but with $x$ approaching an unspecified number, such as $c$? Proceeding as we did in Example 5, we have

$$\lim_{x \to c}(x^2 - 4x) = c \cdot c - 4 \cdot c = c^2 - 4c$$

If we let $f(x) = x^2 - 4x$, we have

$$\lim_{x \to c} f(x) = \lim_{x \to c}(x^2 - 4x) = c^2 - 4c = f(c)$$

That is, this limit can be evaluated simply by evaluating the function $f$ at $c$. It would certainly simplify the process of evaluating limits if we could identify the functions for which

$$\lim_{x \to c} f(x) = f(c) \tag{1}$$

since we could use this fact to evaluate the limit. It turns out that there are many functions that satisfy equation (1). We postpone a detailed discussion of these functions until the next section. For now, we note that if

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0$$

is a polynomial function, then, by the properties in Theorem 1,

$$\begin{aligned}\lim_{x \to c} f(x) &= \lim_{x \to c}(a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0)\\ &= a_n c^n + a_{n-1}c^{n-1} + \cdots + a_0 = f(c)\end{aligned}$$

and if

$$r(x) = \frac{n(x)}{d(x)}$$

is a rational function, where $n(x)$ and $d(x)$ are polynomials with $d(c) \neq 0$, then, by property 7 and the fact that polynomials $n(x)$ and $d(x)$ satisfy equation (1),

$$\lim_{x \to c} r(x) = \lim_{x \to c}\frac{n(x)}{d(x)} = \frac{\lim_{x \to c} n(x)}{\lim_{x \to c} d(x)} = \frac{n(c)}{d(c)} = r(c)$$

These results are summarized in Theorem 3.

**THEOREM 3 LIMITS OF POLYNOMIAL AND RATIONAL FUNCTIONS**

1. $\lim_{x \to c} f(x) = f(c)$ $f$ any polynomial function
2. $\lim_{x \to c} r(x) = r(c)$ $r$ any rational function with a nonzero denominator at $x = c$

**EXAMPLE 6 Evaluating Limits** Find each limit:

(A) $\lim_{x \to 2}(x^3 - 5x - 1)$ (B) $\lim_{x \to -1}\sqrt{2x^2 + 3}$ (C) $\lim_{x \to 4}\frac{2x}{3x + 1}$

**SOLUTION** (A) $\lim_{x \to 2}(x^3 - 5x - 1) = 2^3 - 5 \cdot 2 - 1 = -3$ Theorem 3

(B) $\lim_{x \to -1}\sqrt{2x^2 + 3} = \sqrt{\lim_{x \to -1}(2x^2 + 3)}$ Property 8

$= \sqrt{2(-1)^2 + 3}$ Theorem 3

$= \sqrt{5}$

(C) $\lim_{x \to 4}\frac{2x}{3x + 1} = \frac{2 \cdot 4}{3 \cdot 4 + 1}$ Theorem 3

$= \frac{8}{13}$

**MATCHED PROBLEM 6** Find each limit:

(A) $\lim_{x \to -1}(x^4 - 2x + 3)$ (B) $\lim_{x \to 2}\sqrt{3x^2 - 6}$ (C) $\lim_{x \to -2}\frac{x^2}{x^2 + 1}$

**EXAMPLE 7** **Evaluating Limits** Let

$$f(x) = \begin{cases} x^2 + 1 & \text{if } x < 2 \\ x - 1 & \text{if } x > 2 \end{cases}$$

Find

(A) $\lim_{x \to 2^-} f(x)$ (B) $\lim_{x \to 2^+} f(x)$ (C) $\lim_{x \to 2} f(x)$ (D) $f(2)$

**SOLUTION** (A) $\lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} (x^2 + 1)$ If $x < 2$, $f(x) = x^2 + 1$.

$= 2^2 + 1 = 5$

(B) $\lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} (x - 1)$ If $x > 2$, $f(x) = x - 1$.

$= 2 - 1 = 1$

(C) Since the one-sided limits are not equal, $\lim_{x \to 2} f(x)$ does not exist.

(D) Because the definition of $f$ does not assign a value to $f$ for $x = 2$, only for $x < 2$ and $x > 2$, $f(2)$ does not exist.

**MATCHED PROBLEM 7** Let

$$f(x) = \begin{cases} 2x + 3 & \text{if } x < 5 \\ -x + 12 & \text{if } x > 5 \end{cases}$$

Find each limit:

(A) $\lim_{x \to 5^-} f(x)$ (B) $\lim_{x \to 5^+} f(x)$ (C) $\lim_{x \to 5} f(x)$ (D) $f(5)$

It is important to note that there are restrictions on some of the limit properties. In particular, if

$$\lim_{x \to c} f(x) = 0 \quad \text{and} \quad \lim_{x \to c} g(x) = 0, \quad \text{then finding } \lim_{x \to c} \frac{f(x)}{g(x)}$$

may present some difficulties, since limit property 7 (the limit of a quotient) does not apply when $\lim_{x \to c} g(x) = 0$. The next example illustrates some techniques that can be useful in this situation.

**EXAMPLE 8** **Evaluating Limits** Find each limit:

(A) $\lim_{x \to 2} \frac{x^2 - 4}{x - 2}$

(B) $\lim_{x \to -1} \frac{x|x + 1|}{x + 1}$

**SOLUTION** (A) Algebraic simplification is often useful when the numerator and denominator are both approaching 0.

$$\lim_{x \to 2} \frac{x^2 - 4}{x - 2} = \lim_{x \to 2} \frac{(x - 2)(x + 2)}{x - 2} = \lim_{x \to 2} (x + 2) = 4$$

(B) One-sided limits are helpful for limits involving the absolute value function.

$$\lim_{x \to -1^+} \frac{x|x + 1|}{x + 1} = \lim_{x \to -1^+} (x) = -1 \qquad \text{If } x > -1, \text{ then } \frac{|x + 1|}{x + 1} = 1.$$

$$\lim_{x \to -1^-} \frac{x|x + 1|}{x + 1} = \lim_{x \to -1^-} (-x) = 1 \qquad \text{If } x < -1, \text{ then } \frac{|x + 1|}{x + 1} = -1.$$

Since the limit from the left and the limit from the right are not the same, we conclude that

$$\lim_{x \to -1} \frac{x|x + 1|}{x + 1} \quad \text{does not exist}$$

MATCHED PROBLEM 8 Find each limit:

(A) $\lim_{x \to -3} \frac{x^2 + 4x + 3}{x + 3}$ (B) $\lim_{x \to 4} \frac{x^2 - 16}{|x - 4|}$

**INSIGHT**

In the solution to part A of Example 8, we used the following algebraic identity:

$$\frac{x^2 - 4}{x - 2} = \frac{(x - 2)(x + 2)}{x - 2} = x + 2 \qquad x \neq 2$$

The restriction $x \neq 2$ is necessary here because the first two expressions are not defined at $x = 2$. Why didn't we include this restriction in the solution in part A? When $x$ approaches 2 in a limit problem, it is assumed that $x$ is close, but not equal, to 2. It is important that you understand that both of the following statements are valid:

$$\lim_{x \to 2} \frac{x^2 - 4}{x - 2} = \lim_{x \to 2} (x + 2) \qquad \frac{x^2 - 4}{x - 2} = x + 2, \quad x \neq 2$$

Limits like those in Example 8 occur so frequently in calculus that they are given a special name.

**DEFINITION Indeterminate Form**

If $\lim_{x \to c} f(x) = 0$ and $\lim_{x \to c} g(x) = 0$, then $\lim_{x \to c} \frac{f(x)}{g(x)}$ is said to be **indeterminate,** or, more specifically, a **0/0 indeterminate form.**

The term *indeterminate* is used because the limit of an indeterminate form may or may not exist (see parts A and B of Example 8).

⚠ CAUTION The expression 0/0 does not represent a real number and should never be used as the value of a limit. If a limit is a 0/0 indeterminate form, further investigation is always required to determine whether the limit exists and to find its value if it does exist.

If the denominator of a quotient approaches 0 and the numerator approaches a nonzero number, the limit of the quotient is not an indeterminate form. In fact, a limit of this form never exists, as Theorem 4 states.

**THEOREM 4 LIMIT OF A QUOTIENT**

If $\lim_{x \to c} f(x) = L$, $L \neq 0$, and $\lim_{x \to c} g(x) = 0$,

then

$$\lim_{x \to c} \frac{f(x)}{g(x)} \quad \text{does not exist.}$$

*Explore & Discuss* **2**

Use algebraic and/or graphical techniques to analyze each of the following indeterminate forms:

(A) $\lim_{x\to 1} \frac{x-1}{x^2-1}$ (B) $\lim_{x\to 1} \frac{(x-1)^2}{x^2-1}$ (C) $\lim_{x\to 1} \frac{x^2-1}{(x-1)^2}$

## Limits of Difference Quotients

Let the function $f$ be defined in an open interval containing the number $a$. One of the most important limits in calculus is the limit of the **difference quotient:**

$$\lim_{h\to 0} \frac{f(a+h)-f(a)}{h} \tag{3}$$

If

$$\lim_{h\to 0} [f(a+h)-f(a)] = 0$$

as it often does, then limit (3) is an indeterminate form. The examples that follow illustrate some useful techniques for evaluating limits of difference quotients.

**EXAMPLE 9** **Limit of a Difference Quotient** Find the following limit for $f(x) = 4x - 5$:

$$\lim_{h\to 0} \frac{f(3+h)-f(3)}{h}$$

**SOLUTION**

$$\lim_{h\to 0} \frac{f(3+h)-f(3)}{h} = \lim_{h\to 0} \frac{[4(\mathbf{3}+\boldsymbol{h})-5]-[4(\mathbf{3})-5]}{h}$$

$$= \lim_{h\to 0} \frac{12+4h-5-12+5}{h}$$

$$= \lim_{h\to 0} \frac{4h}{h} = \lim_{h\to 0} 4 = 4$$

*Since this is a 0/0 indeterminate form and property 7 in Theorem 2 does not apply, we proceed with algebraic simplification.*

**MATCHED PROBLEM 9** Find the following limit for $f(x) = 7 - 2x$: $\lim_{h\to 0} \frac{f(4+h)-f(4)}{h}$.

*Explore & Discuss* **3**

The following is an incorrect solution to Example 9, with the invalid statements indicated by $\neq$:

$$\lim_{h\to 0} \frac{f(3+h)-f(3)}{h} \neq \lim_{h\to 0} \frac{4(3+h)-5-4(3)-5}{h}$$

$$= \lim_{h\to 0} \frac{-10+4h}{h}$$

$$\neq \lim_{h\to 0} \frac{-10+4}{1} = -6$$

Explain why each $\neq$ is used.

**EXAMPLE 10** **Limit of a Difference Quotient** Find the following limit for $f(x) = |x+5|$:

$$\lim_{h\to 0} \frac{f(-5+h)-f(-5)}{h}$$

**SOLUTION**

$$\lim_{h\to 0}\frac{f(-5+h)-f(-5)}{h} = \lim_{h\to 0}\frac{|(-5+h)+5| - |-5+5|}{h}$$

$$= \lim_{h\to 0}\frac{|h|}{h} \quad \text{does not exist}$$

*Since this is a 0/0 indetermi nate form and property 7 in Theorem 2 does not apply, we proceed with algebraic simplification.*

**MATCHED PROBLEM 10** Find the following limit for $f(x) = |x - 1|$: $\lim_{h\to 0}\frac{f(1+h)-f(1)}{h}$.

**EXAMPLE 11** **Limit of a Difference Quotient** Find the following limit for $f(x) = \sqrt{x}$:

$$\lim_{h\to 0}\frac{f(2+h)-f(2)}{h}$$

**SOLUTION**

$$\lim_{h\to 0}\frac{f(2+h)-f(2)}{h} = \lim_{h\to 0}\frac{\sqrt{2+h}-\sqrt{2}}{h}$$

*This is a 0/0 indeterminate form, so property 7 in Theorem 2 does not apply. Rationalizing the numerator will be of help.*

$$= \lim_{h\to 0}\frac{\sqrt{2+h}-\sqrt{2}}{h}\cdot\frac{\sqrt{2+h}+\sqrt{2}}{\sqrt{2+h}+\sqrt{2}}$$

*$(A - B)(A + B) = A^2 - B^2$*

$$= \lim_{h\to 0}\frac{2+h-2}{h(\sqrt{2+h}+\sqrt{2})}$$

$$= \lim_{h\to 0}\frac{1}{\sqrt{2+h}+\sqrt{2}}$$

$$= \frac{1}{\sqrt{2}+\sqrt{2}} = \frac{1}{2\sqrt{2}}$$

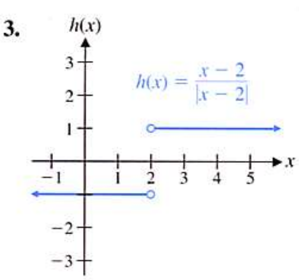
**MATCHED PROBLEM 11** Find the following limit for $f(x) = \sqrt{x}$: $\lim_{h\to 0}\frac{f(3+h)-f(3)}{h}$.

*Answers to Matched Problem*

1.

| $x$ | −2 | −1 | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|---|---|
| $h(x)$ | 1.0 | 1.5 | 2.0 | 2.5 | 3.0 | 3.5 | 4.0 |

2. $\lim_{x\to 1} f(x) = 2$

3. 


$\lim_{x\to 2}\frac{x-2}{|x-2|}$ does not exist

4. (A) $\lim_{x\to 0^-} f(x) = 0$
$\lim_{x\to 0^+} f(x) = 0$
$\lim_{x\to 0} f(x) = 0$
$f(0) = 0$

(B) $\lim_{x\to 1^-} f(x) = 1$
$\lim_{x\to 1^+} f(x) = 2$
$\lim_{x\to 1} f(x)$ does not exist
$f(1)$ not defined

(C) $\lim_{x \to 3^-} f(x) = 3$
$\lim_{x \to 3^+} f(x) = 3$
$\lim_{x \to 3} f(x) = 3$
$f(3)$ not defined

5. $-6$

6. (A) 6
(B) $\sqrt{6}$
(C) $\frac{4}{5}$

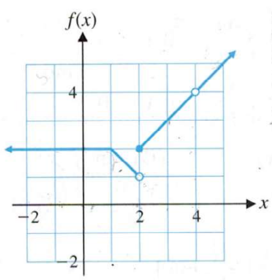
7. (A) 13
(B) 7
(C) Does not exist
(D) Not defined

8. (A) $-2$
(B) Does not exist

9. $-2$ 10. Does not exist 11. $1/(2\sqrt{3})$

## Exercise 10-1

*In Problems 1–4, use the graph of the function f shown to estimate the indicated limits and function values.*



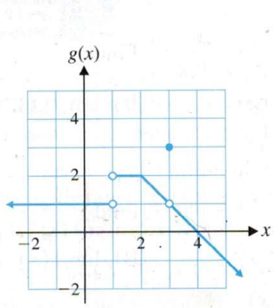

1. (A) $\lim_{x \to 0^-} f(x)$ (B) $\lim_{x \to 0^+} f(x)$
(C) $\lim_{x \to 0} f(x)$ (D) $f(0)$

2. (A) $\lim_{x \to 1^-} f(x)$ (B) $\lim_{x \to 1^+} f(x)$
(C) $\lim_{x \to 1} f(x)$ (D) $f(1)$

3. (A) $\lim_{x \to 2^-} f(x)$ (B) $\lim_{x \to 2^+} f(x)$
(C) $\lim_{x \to 2} f(x)$ (D) $f(2)$
(E) Is it possible to redefine $f(2)$ so that $\lim_{x \to 2} f(x) = f(2)$? Explain.

4. (A) $\lim_{x \to 4^-} f(x)$ (B) $\lim_{x \to 4^+} f(x)$
(C) $\lim_{x \to 4} f(x)$ (D) $f(4)$
(E) Is it possible to define $f(4)$ so that $\lim_{x \to 4} f(x) = f(4)$? Explain.

*In Problems 5–8, use the graph of the function g shown to estimate the indicated limits and function values.*



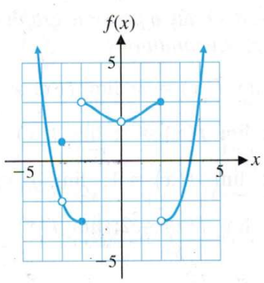

5. (A) $\lim_{x \to 1^-} g(x)$ (B) $\lim_{x \to 1^+} g(x)$
(C) $\lim_{x \to 1} g(x)$ (D) $g(1)$
(E) Is it possible to define $g(1)$ so that $\lim_{x \to 1} g(x) = g(1)$? Explain.

6. (A) $\lim_{x \to 2^-} g(x)$ (B) $\lim_{x \to 2^+} g(x)$
(C) $\lim_{x \to 2} g(x)$ (D) $g(2)$

7. (A) $\lim_{x \to 3^-} g(x)$ (B) $\lim_{x \to 3^+} g(x)$
(C) $\lim_{x \to 3} g(x)$ (D) $g(3)$
(E) Is it possible to redefine $g(3)$ so that $\lim_{x \to 3} g(x) = g(3)$? Explain.

8. (A) $\lim_{x \to 4^-} g(x)$ (B) $\lim_{x \to 4^+} g(x)$
(C) $\lim_{x \to 4} g(x)$ (D) $g(4)$

*In Problems 9–12, use the graph of the function f shown to estimate the indicated limits and function values.*




9. (A) $\lim_{x \to -3^+} f(x)$ (B) $\lim_{x \to -3^-} f(x)$
(C) $\lim_{x \to -3} f(x)$ (D) $f(-3)$
(E) Is it possible to redefine $f(-3)$ so that $\lim_{x \to -3} f(x) = f(-3)$? Explain.

10. (A) $\lim_{x \to -2^+} f(x)$ (B) $\lim_{x \to -2^-} f(x)$
(C) $\lim_{x \to -2} f(x)$ (D) $f(-2)$
(E) Is it possible to define $f(-2)$ so that $\lim_{x \to -2} f(x) = f(-2)$? Explain.

**11.** (A) $\lim_{x\to 0^+} f(x)$ (B) $\lim_{x\to 0^-} f(x)$
(C) $\lim_{x\to 0} f(x)$ (D) $f(0)$
(E) Is it possible to redefine $f(0)$ so that $\lim_{x\to 0} f(x) = f(0)$? Explain.

**12.** (A) $\lim_{x\to 2^+} f(x)$ (B) $\lim_{x\to 2^-} f(x)$
(C) $\lim_{x\to 2} f(x)$ (D) $f(2)$
(E) Is it possible to redefine $f(2)$ so that $\lim_{x\to 2} f(x) = f(2)$? Explain.

*In Problems 13–22, find each limit if it exists.*

**13.** $\lim_{x\to 3} 4x$
**14.** $\lim_{x\to -2} 3x$
**15.** $\lim_{x\to -4} (x + 5)$
**16.** $\lim_{x\to 5} (x - 3)$
**17.** $\lim_{x\to 2} x(x - 4)$
**18.** $\lim_{x\to -1} x(x + 3)$
**19.** $\lim_{x\to -3} \dfrac{x}{x + 5}$
**20.** $\lim_{x\to 4} \dfrac{x - 2}{x}$
**21.** $\lim_{x\to 1} \sqrt{5x + 4}$
**22.** $\lim_{x\to 0} \sqrt{16 - 7x}$

*Given that $\lim_{x\to 1} f(x) = -5$ and $\lim_{x\to 1} g(x) = 4$, find the indicated limits in Problems 23–34.*

**23.** $\lim_{x\to 1} (-3)f(x)$
**24.** $\lim_{x\to 1} 2g(x)$
**25.** $\lim_{x\to 1} [2f(x) + g(x)]$
**26.** $\lim_{x\to 1} [g(x) - 3f(x)]$
**27.** $\lim_{x\to 1} \dfrac{2 - f(x)}{x + g(x)}$
**28.** $\lim_{x\to 1} \dfrac{3 - f(x)}{1 - 4g(x)}$
**29.** $\lim_{x\to 1} f(x)[2 - g(x)]$
**30.** $\lim_{x\to 1} [f(x) - 7x]g(x)$
**31.** $\lim_{x\to 1} \sqrt{g(x) - f(x)}$
**32.** $\lim_{x\to 1} \sqrt[3]{2x + 2f(x)}$
**33.** $\lim_{x\to 1} [f(x) + 1]^2$
**34.** $\lim_{x\to 1} [2 - g(x)]^3$

*In Problems 35–38, sketch a possible graph of a function that satisfies the given conditions.*

**35.** $f(0) = 1$; $\lim_{x\to 0^-} f(x) = 3$; $\lim_{x\to 0^+} f(x) = 1$

**36.** $f(1) = -2$; $\lim_{x\to 1^-} f(x) = 2$; $\lim_{x\to 1^+} f(x) = -2$

**37.** $f(-2) = 2$; $\lim_{x\to -2^-} f(x) = 1$; $\lim_{x\to -2^+} f(x) = 1$

**38.** $f(0) = -1$; $\lim_{x\to 0^-} f(x) = 2$; $\lim_{x\to 0^+} f(x) = 2$

## B In Problems 39–54, find each indicated quantity if it exists.

**39.** Let $f(x) = \begin{cases} 1 - x^2 & \text{if } x \le 0 \\ 1 + x^2 & \text{if } x > 0 \end{cases}$. Find
(A) $\lim_{x\to 0^+} f(x)$ (B) $\lim_{x\to 0^-} f(x)$
(C) $\lim_{x\to 0} f(x)$ (D) $f(0)$

**40.** Let $f(x) = \begin{cases} 2 + x & \text{if } x \le 0 \\ 2 - x & \text{if } x > 0 \end{cases}$. Find
(A) $\lim_{x\to 0^+} f(x)$ (B) $\lim_{x\to 0^-} f(x)$
(C) $\lim_{x\to 0} f(x)$ (D) $f(0)$

**41.** Let $f(x) = \begin{cases} x^2 & \text{if } x < 1 \\ 2x & \text{if } x > 1 \end{cases}$. Find
(A) $\lim_{x\to 1^+} f(x)$ (B) $\lim_{x\to 1^-} f(x)$
(C) $\lim_{x\to 1} f(x)$ (D) $f(1)$

**42.** Let $f(x) = \begin{cases} x + 3 & \text{if } x < -2 \\ \sqrt{x + 2} & \text{if } x > -2 \end{cases}$. Find
(A) $\lim_{x\to -2^+} f(x)$ (B) $\lim_{x\to -2^-} f(x)$
(C) $\lim_{x\to -2} f(x)$ (D) $f(-2)$

**43.** Let $f(x) = \begin{cases} \dfrac{x^2 - 9}{x + 3} & \text{if } x < 0 \\ \dfrac{x^2 - 9}{x - 3} & \text{if } x > 0 \end{cases}$. Find
(A) $\lim_{x\to -3} f(x)$ (B) $\lim_{x\to 0} f(x)$
(C) $\lim_{x\to 3} f(x)$

**44.** Let $f(x) = \begin{cases} \dfrac{x}{x + 3} & \text{if } x < 0 \\ \dfrac{x}{x - 3} & \text{if } x > 0 \end{cases}$. Find
(A) $\lim_{x\to -3} f(x)$ (B) $\lim_{x\to 0} f(x)$
(C) $\lim_{x\to 3} f(x)$

**45.** Let $f(x) = \dfrac{|x - 1|}{x - 1}$. Find
(A) $\lim_{x\to 1^+} f(x)$ (B) $\lim_{x\to 1^-} f(x)$
(C) $\lim_{x\to 1} f(x)$ (D) $f(1)$

**46.** Let $f(x) = \dfrac{x - 3}{|x - 3|}$. Find
(A) $\lim_{x\to 3^+} f(x)$ (B) $\lim_{x\to 3^-} f(x)$
(C) $\lim_{x\to 3} f(x)$ (D) $f(3)$

**47.** Let $f(x) = \dfrac{x - 2}{x^2 - 2x}$. Find
(A) $\lim_{x\to 0} f(x)$ (B) $\lim_{x\to 2} f(x)$
(C) $\lim_{x\to 4} f(x)$

**48.** Let $f(x) = \dfrac{x + 3}{x^2 + 3x}$. Find
(A) $\lim_{x\to -3} f(x)$ (B) $\lim_{x\to 0} f(x)$
(C) $\lim_{x\to 3} f(x)$

**49.** Let $f(x) = \dfrac{x^2 - x - 6}{x + 2}$. Find
(A) $\lim_{x\to -2} f(x)$ (B) $\lim_{x\to 0} f(x)$
(C) $\lim_{x\to 3} f(x)$

**50.** Let $f(x) = \dfrac{x^2 + x - 6}{x + 3}$. Find
(A) $\lim_{x\to -3} f(x)$ (B) $\lim_{x\to 0} f(x)$
(C) $\lim_{x\to 2} f(x)$